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Existence of a tree of Stieltjes strings corresponding to two given spectra

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Abstract

Transversal vibrations of a plane rooted tree of Stieltjes strings are considered with Dirichlet boundary conditions at each pendant vertex. Continuity and Kirchhoff conditions are imposed at each interior vertex except the root. The Dirichlet problem is that with the Dirichlet condition at the root, and the Neumann problem is that with continuity and Kirchhoff conditions at the root. It is shown that strict interlacing is a sufficient condition for two sequences of real numbers to be the spectra of Neumann and Dirichlet problems generated by Stieltjes string recurrent relations on any prescribed tree.

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1. Introduction

Inverse spectral and scattering problems on graph domains are a rapidly developing branch of analysis. Usually the Sturm–Liouville, the Dirac equation or the string equation is considered on the edges of a graph subject to matching and boundary conditions at the vertices. This can be described as a spectral problem for an operator (see e.g. [9, 10, 30]) which is self-adjoint under certain conditions [6, 20]. The justification for such models can be found in [22]. There are different approaches to inverse problems on graphs: (1) recovering the form of a metric graph using spectral or scattering data (see e.g. [1, 16]), (2) the form of the graph is known *a priori* and the aim is to recover the potentials of the Sturm–Liouville equation on the edges (see e.g. [15, 23, 28, 29, 32]) or the mass distribution [3, 27] using spectral or scattering data.

The tradition of using Dirichlet and Neumann spectra to solve an inverse problem on a graph [5, 32] came from the classical results of Borg [2] and Levitan and Gasymov [26] for the inverse spectral problem on a finite interval. It is possible to consider two boundary value problems with the Neumann condition at a certain pendant vertex and with the Dirichlet condition at this vertex [5, 32] and to use their spectra as the given data for solving the inverse problem.

However, in the case of graph domains it is possible [25] to introduce analogues of Neumann and Dirichlet conditions at an interior vertex chosen as the root. Namely, for the case of Sturm–Liouville equations, $-y_j'' + q_j(x)y_j = \lambda^2 y_j$ on the edges $y_j(a_j) = 0$ $(j = 1, 2, \ldots, d(\mathbf{v}))$ where $\mathbf{v}$ is an interior vertex (the root) of degree $d(\mathbf{v})$ corresponds to Dirichlet conditions and $y_j(a_j) = y_k(a_k)$ and $\sum_{j=1}^{d(\mathbf{v})} y_j'(a_j) = 0$ corresponds to the Neumann case. In such terms, the problems in [3, 14, 17, 28, 29] where star graphs were considered can be treated as Neumann–Dirichlet two-spectral problems. Here we extend the results of [3] and [4] to the case of a tree, since we are interested in the existence of the solution of the inverse problem admitting the non-uniqueness of it.

Let us recall that Krein ([21], see also [13]) called a massless thread bearing beads (point masses) a Stieltjes string. Such strings are widely used as a simple model in mechanics [11, 12, 24], and one finds a similar equation in the theory of electric circuits [31] (the Cauer method [7, 31]). For the simple case of one string, the inverse problem for two spectra was completely solved in [13]. In particular, in [13] the procedure of recovering the values of masses and intervals between them by the two spectra and the total length of the string was given as well as characterization of the spectra. In [8], the masses and the intervals between them were calculated using frequencies of vibrations obtained from experiment by the method used in [13].

We consider the recurrent relations of the Stieltjes string defined on a tree domain. An arbitrary vertex $\mathbf{v}$ is chosen as the root. We impose continuity and Kirchhoff conditions at all the interior vertices except at $\mathbf{v}$. At the pendant vertices, Dirichlet conditions are satisfied. At the root $\mathbf{v}$, we impose continuity conditions if the root is not pendant. Then we consider the Dirichlet problem, which we obtain by adding the Dirichlet condition at the root, and the Neumann problem, where we add the Kirchhoff condition at the root. If the root is pendant then the Kirchhoff condition is nothing but the Neumann condition. If the total number of the point masses is n then the Dirichlet problem has $2n$ eigenvalues, and the same is true for the Neumann problem. These eigenvalues interlace. In general, this interlacing is not strict:

$$0 < \mu_1^2 \leqslant \nu_1^2 \leqslant \cdots \leqslant \mu_{n-1}^2 \leqslant \nu_{n-1}^2 \leqslant \mu_n^2 \leqslant \nu_n^2 (\mu_{-k} = -\mu_k, \nu_{-k} = -\nu_k).$$

Characterization of the two spectra for a star-shaped graph rooted at the interior vertex has been given in [3]. Here, we establish necessary conditions for two sequences of real numbers to be the spectra of the Neumann and Dirichlet problems on an arbitrary tree. These conditions include the above-mentioned interlacing and the inequalities: (1) $p_N(z) \leqslant q - \kappa$, where $p_N(z)$ is the multiplicity of $\lambda = \sqrt{z}$ as for an eigenvalue of the Neumann problem, q is the number of the edges, κ is the number of the interior vertices; (2) for an interior vertex $p_D(z) \leqslant q - \kappa + 1$ where $p_D(z)$ is the multiplicity of $\lambda = \sqrt{z}$ as for an eigenvalue of the Dirichlet problem and (3) for a pendant vertex $p_D(z) + p_N(z) \leqslant 2q - 2\kappa - 1$.

It is shown that, for any two strictly interlacing finite sets of real numbers,

$$0 < \mu_1^2 < \cdots < \mu_{n-1}^2 < \nu_{n-1}^2 < \mu_n^2 < \nu_n^2,$$

and for any finite metric tree, a suitable placement of point masses on the tree will generate these prescribed spectra $\{\mu_k\}_{k=-n,k\neq 0}^{n}$ and $\{\nu_k\}_{k=-n,k\neq 0}^{n}$ $(\mu_{-k} = -\mu_k, \nu_{-k} = -\nu_k)$.

2. Characteristic polynomials

Let T be a plane metric tree with q edges. We denote by v_j the vertices, by $d(v_j)$ their degrees, by e_j their edges and by l_j their lengths. An arbitrary vertex $\mathbf{v}$ is chosen to be the root. Local coordinates for edges identify the edge e_j with the interval $[0, l_j]$ so that the local coordinate increases as the distance to the root decreases. Each edge e_j is divided into $n_j + 1$ subintervals of lengths $l_j^0, l_j^1, \ldots, l_j^{n_j}$ by point masses $m_j^1, m_j^2, \ldots, m_j^{n_j}$ $\left(l_j^k > 0, m_j^k > 0, l_j = \sum_{k=0}^{n_j} l_j^k\right)$. The masses and the intervals are enumerated such that the upper index increases as the distance to the root decreases. This means that each pendant vertex (if it is not **v**) is located at the beginning of some interval of length l_j^0. The root is at the end of a subinterval of length $l_j^{n_j}$ on each edge e_j incident with **v**. All the other interior vertices v have one outgoing edge e_j starting with a subinterval of length l_j^0, while each incoming edge e_r ends at v with an interval of lengths $l_r^{n_r}$. It is assumed that the tree is stretched and the pendant vertices are fixed. The tree can vibrate in the direction orthogonal to the equilibrium position of the strings. The transverse displacement of the mass m_j^k is denoted by $w_j^k(t)$. If an edge e_j is incoming for an interior vertex v then the displacement of the incoming end of the edge is denoted by $w_j^{n_j+1}(t)$, while if an edge e_r is outgoing for a vertex v then the displacement of the outgoing end of the edge is denoted by $w_r^0(t)$. Using such notation, vibrations of the graph can be described by the system of equations

$$\frac{w_j^k(t) - w_j^{k+1}(t)}{l_j^k} + \frac{w_j^k(t) - w_j^{k-1}(t)}{l_j^{k-1}} + m_j^k \frac{\partial^2 w_j^k}{\partial t^2}(t) = 0$$
$$(k = 1, 2, \ldots, n_j; n_j \geqslant 1, j = 1, 2, \ldots, q). \tag{2.1}$$

For each interior vertex (except the root) with incoming edges e_j and outgoing edge e_r we impose the continuity conditions

$$w_r^0(t) = w_j^{n_j+1}(t) \tag{2.2}$$

for all j corresponding to incoming edges. The balance of forces at such a vertex implies

$$\frac{w_r^1(t) - w_r^0(t)}{l_r^0} = \sum_j \frac{w_j^{n_j+1}(t) - w_j^{n_j}(t)}{l_j^{n_j}}, \tag{2.3}$$

where the sum on the right-hand side is taken over all the incoming edges. For an edge e_j incident with a pendant vertex (except the root if it is pendant), we impose the Dirichlet boundary condition

$$w_j^0(t) = 0. \tag{2.4}$$

The continuity conditions at the root are

$$w_r^{n_r+1}(t) = w_j^{n_j+1}(t) \tag{2.5}$$

for all pairs of edges incident with **v**. We need to impose one more condition at the root. We consider two cases: the Dirichlet case with

$$w_j^{n_j+1}(t) = 0 \tag{2.6}$$

and the Neumann case

$$\sum_j \frac{w_j^{n_j+1}(t) - w_j^{n_j}(t)}{l_j^{n_j}} = 0, \tag{2.7}$$

where the sum is taken over all the edges incident with the root.

Substituting $w_j^k(t) = \mathrm{e}^{\mathrm{i}\lambda t} u_j^k$ into (2.1)–(2.7) we obtain the Dirichlet problem described below.

For each edge:

$$\frac{u_j^k - u_j^{k+1}}{l_j^k} + \frac{u_j^k - u_j^{k-1}}{l_j^{k-1}} - m_j^k \lambda^2 u_j^k = 0 \qquad (k = 1, 2, \ldots, n_j, j = 1, 2, \ldots, q). \tag{2.8}$$

For each interior vertex (except of the root) with incoming edges e_j and outgoing edge e_r we have

$$u_r^0 = u_j^{n_j+1}. \tag{2.9}$$

$$\frac{u_r^1 - u_r^0}{l_r^0} = \sum_j \frac{u_j^{n_j+1} - u_j^{n_j}}{l_j^{n_j}}. \tag{2.10}$$

For each edge e_j incident with a pendant vertex (except the root if it is pendant)

$$u_j^0 = 0. \tag{2.11}$$

At the root **v**:

$$u_j^{n_j+1} = 0 \tag{2.12}$$

for all edges incident with **v**.

The conditions

$$u_k^{n_k+1} = u_j^{n_j+1} \tag{2.13}$$

for all pairs of edges e_k and e_j incident with the root together with

$$\sum_j \frac{u_j^{n_j+1} - u_j^{n_j}}{l_j^{n_j}} = 0 \tag{2.14}$$

we call Neumann conditions at the root. If the root is a pendant vertex then (2.13), (2.14) are equivalent to the usual Neumann condition. In what follows, problem (2.8)–(2.12) is called the Dirichlet problem for the tree T and problem (2.8)–(2.11), (2.13), (2.14) is called the Neumann problem.

The fundamental system of two linearly independent solutions for (2.8) can be composed by the polynomials (see [13]) $R_{2k-2}^{(j)}(\lambda^2, l_j^0)$ and $R_{2k-2}^{(j)}(\lambda^2, \infty)$ which satisfy the initial conditions $R_0^{(j)}(\lambda^2, l_j^0) = 1$, $R_{-1}^{(j)}(\lambda^2, l_j^0) = \frac{1}{l_j^0}$, $R_0^{(j)}(\lambda^2, \infty) = 1$, $R_{-1}^{(j)}(\lambda^2, \infty) = 0$.

We are looking for the solutions of (2.8) in the form $u_{(j)}^k = R_{2k-2}^{(j)}(\lambda^2, l_j^0) g_j^1 + R_{2k-2}^{(j)}(\lambda^2, \infty) h_j^1$ on the edge e_j and introduce the polynomials of odd index:

$$R_{2k-1}^{(j)}(\lambda^2, l_j^0) = \frac{R_{2k}^{(j)}(\lambda^2, l_j^0) - R_{2k-2}^{(j)}(\lambda^2, l_j^0)}{l_j^k},$$

$$R_{2k-1}^{(j)}(\lambda^2, \infty) = \frac{R_{2k}^{(j)}(\lambda^2, \infty) - R_{2k-2}^{(j)}(\lambda^2, \infty)}{l_j^k}.$$

Then these polynomials satisfy the relations [13]

$$\begin{aligned} R_{2k-1}^{(j)}(\lambda^2, l_j^0) &= -\lambda^2 m_k^{(j)} R_{2k-2}^{(j)}(\lambda^2, l_j^0) + R_{2k-3}^{(j)}(\lambda^2, l_j^0), \\ R_{2k-1}^{(j)}(\lambda^2, \infty) &= -\lambda^2 m_k^{(j)} R_{2k-2}^{(j)}(\lambda^2, \infty) + R_{2k-3}^{(j)}(\lambda^2, \infty), \end{aligned} \tag{2.15}$$

$$\begin{aligned} R_{2k}^{(j)}(\lambda^2, l_j^0) &= l_k^{(j)} R_{2k-1}^{(j)}(\lambda^2, l_j^0) + R_{2k-2}^{(j)}(\lambda^2, l_j^0) \qquad (k = 1, 2, \ldots, n_j), \\ R_{2k}^{(j)}(\lambda^2, \infty) &= l_k^{(j)} R_{2k-1}^{(j)}(\lambda^2, \infty) + R_{2k-2}^{(j)}(\lambda^2, \infty) \qquad (k = 1, 2, \ldots, n_j). \end{aligned} \tag{2.16}$$

With this notation we obtain the following using (2.9)–(2.12).

For each interior vertex (except the root) with incoming edges e_j and outgoing edge e_k:

$$R_{2n_j}^{(j)}\left(\lambda^2, l_j^0\right)g_j^1 + R_{2n_j}^{(j)}(\lambda^2, \infty)h_j^1 = h_k^1, \tag{2.17}$$

$$\frac{g_k^1}{l_k^0} = \sum_j \left(R_{2n_j-1}^{(j)}\left(\lambda^2, l_j^0\right)g_j^1 + R_{2n_j-1}^{(j)}(\lambda^2, \infty)h_j^1\right). \tag{2.18}$$

For each edge e_j incident with a pendant vertex (except the root if it is pendant)

$$h_j^1 = 0. \tag{2.19}$$

At the root $\mathbf{v}$ we have

$$R_{2n_j-1}^{(j)}\left(\lambda^2, l_j^0\right)g_j^1 + R_{2n_j-1}^{(j)}(\lambda^2, \infty)h_j^1 = 0 \tag{2.20}$$

for all the edges incident with $\mathbf{v}$. Equations (2.17)–(2.20) complete the Dirichlet problem, while (2.17)–(2.19) together with equations

$$R_{2n_j}^{(j)}\left(\lambda^2, l_j^0\right)g_j^1 + R_{2n_j}^{(j)}(\lambda^2, \infty)h_j^1 = R_{2n_k}^{(j)}\left(\lambda^2, l_k^0\right)g_k^1 + R_{2n_k}^{(j)}(\lambda^2, \infty)h_k^1 \tag{2.21}$$

and

$$\sum_j \left(R_{2n_j-1}^{(j)}\left(\lambda^2, l_j^0\right)g_j^1 + R_{2n_j-1}^{(j)}(\lambda^2, \infty)h_k^1\right) = 0 \tag{2.22}$$

complete the Neumann problem.

Then the *characteristic function* of the Neumann problem, i.e., a polynomial whose zeros coincide with the spectrum of the problem, can be expressed by $l_j^0 R_{2n_j}^{(j)}\left(\lambda^2, l_j^0\right)$, $l_j^0 R_{2n_j-1}^{(j)}\left(\lambda^2, l_j^0\right)$, $R_{2n_j}^{(j)}(\lambda^2, \infty)$ and $R_{2n_j-1}^{(j)}(\lambda^2, \infty)$. To do this, we introduce the following system of vectors:

$$\psi_j(\lambda^2) = \operatorname{col}\Big\{\underbrace{0, 0, \ldots, 0, l_j^0 R_{-2}^{(j)}\left(\lambda^2, l_j^0\right), l_j^0 R_0^{(j)}\left(\lambda^2, l_j^0\right), \ldots, l_j^0 R_{2n_j}^{(j)}\left(\lambda^2, l_j^0\right), 0, 0, \ldots, 0,}_{n+2q}$$
$$\underbrace{0, 0, \ldots, 0}_{n+2q}\Big\},$$
$$\psi_{j+q}(\lambda^2) = \operatorname{col}\Big\{\underbrace{0, 0, \ldots, 0,}_{n+2q}$$
$$\underbrace{0, 0, \ldots, 0, R_{-2}^{(j)}(\lambda^2, \infty), R_0^{(j)}(\lambda^2, \infty), \ldots, R_{2n_j}^{(j)}(\lambda^2, \infty), 0, 0, \ldots, 0}_{n+2q}\Big\}$$

for $j = 1, 2, \ldots, q$, where q is the number of edges in T, $n = \sum_{j=1}^q n_j$. As in [30] we denote by L_j $(j = 1, 2, \ldots, 2q)$ the linear functionals $C^{2n+4q} \to C$ generated by (2.17)–(2.22). Then $\Phi(\lambda^2) = \{L_j(\psi_p(\lambda^2)\}_{j,p}^{2q}$ is the characteristic matrix which represents the system of linear equations describing the boundary conditions at pendant vertices and continuity and Kirchhoff conditions for the interior vertices. We call

$$\phi_N(\lambda^2) := \det(\Phi(\lambda^2))$$

the *characteristic function* of problem (2.8)–(2.11), (2.13), (2.14). For the sake of convenience, we use the spectral parameter $z = \lambda^2$. It is easy to see that the characteristic function satisfies

$$\phi_N(\overline{z}) = \overline{\phi_N(z)}.$$

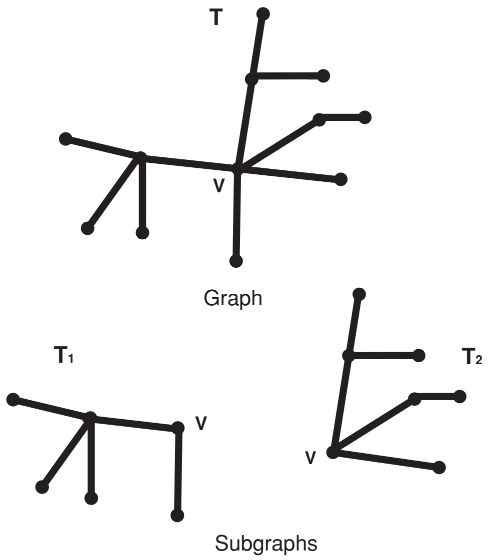

Figure 1.

We are also interested in the problem generated by the same equations and the same boundary and matching conditions, but with conditions (2.12) instead of (2.13), (2.14) at **v**. We denote this characteristic function of problem (2.8)–(2.12) by $\phi_D(\lambda^2)$. In the case when **v** is a pendant vertex condition, (2.12) coincides with the Dirichlet boundary condition. Also $\phi_D(z)$ satisfies the symmetry condition

$$\phi_D(\overline{z}) = \overline{\phi_D(z)}.$$

Let us assume that the root **v** is an interior vertex. We divide our tree T into two subtrees T_1 and T_2 having **v** as the only common vertex. (We say that T_1 and T_2 are the *complementary subtrees* of T; see figure 1.)

We consider two Neumann problems on the subtrees generated by the recurrence relations on the edges,

$$\frac{u_{j,s}^k - u_{j,s}^{k+1}}{l_{j,s}^k} + \frac{u_{j,s}^k - u_{j,s}^{k-1}}{l_{j,s}^{k-1}} - m_{j,s}^k \lambda^2 u_{j,s}^k = 0 \qquad (k = 1, 2, \ldots, n_{j,s}, s = 1, 2). \tag{2.23}$$

For an edge $e_{j,s} \in T_s$ incident with a pendant vertex, we impose the Dirichlet boundary condition

$$u_{j,s}^0 = 0. \tag{2.24}$$

For each interior vertex of T_s (except the root) with incoming edges $e_{j,s}$ and outgoing edge $e_{r,s}$, we have

$$u_{r,s}^0 = u_{j,s}^{n_{j,s}+1}, \tag{2.25}$$

$$\frac{u_{r,s}^1 - u_{r,s}^0}{l_{r,s}^0} = \sum_j \frac{u_{j,s}^{n_{j,s}+1} - u_{j,s}^{n_{j,s}}}{l_{j,s}^{n_{j,s}}}. \tag{2.26}$$

At the root **v** of the subtree T_s, we still have the continuity conditions

$$u_{r,s}^{n_{r,s}+1} = u_{j,s}^{n_{j,s}+1} \tag{2.27}$$

for all pairs of edges of T_s incident with $\mathbf{v}$ and the Kirchhoff condition

$$\sum_j \frac{u_{j,s}^{n_{j,s}+1} - u_{j,s}^{n_{j,s}}}{l_{j,s}^{n_{j,s}}} = 0. \tag{2.28}$$

Here the sum is taken over all the edges of T_s incident with $\mathbf{v}$.

The two Dirichlet problems on T_1 and T_2 are described as follows:

$$\frac{u_{j,s}^k - u_{j,s}^{k+1}}{l_{j,s}^k} + \frac{u_{j,s}^k - u_{j,s}^{k-1}}{l_{j,s}^{k-1}} - m_{j,s}^k \lambda^2 u_{j,s}^k = 0 \qquad (k = 1, 2, \ldots, n_{j,s}, s = 1, 2); \tag{2.29}$$

for an edge $e_{j,s} \in T_s$ incident with a pendant vertex we have

$$u_{j,s}^0 = 0, \tag{2.30}$$

for each interior vertex of T_s (except the root) with incoming edges $e_{j,s}$ and outgoing edge $e_{r,s}$:

$$u_{r,s}^0 = u_{j,s}^{n_{j,s}+1}, \tag{2.31}$$

$$\frac{u_{r,s}^1 - u_{r,s}^0}{l_{r,s}^0} = \sum_j \frac{u_{j,s}^{n_{j,s}+1} - u_{j,s}^{n_{j,s}}}{l_{j,s}^{n_{j,s}}}, \tag{2.32}$$

and at the root $\mathbf{v}$:

$$u_{r,s}^{n_{r,s}+1} = 0 \tag{2.33}$$

for all edges of T_s incident with $\mathbf{v}$.

Denote by $\phi_N^{(s)}(z)$ the characteristic function of problems (2.23)–(2.28) and by $\phi_D^{(s)}(z)$ the characteristic function of problems (2.29)–(2.33) with $z = \lambda^2$. With these terminologies, we obtain a reduction formula for the characteristic functions of a tree. The following theorem is a discrete analogue of theorem 2.1 in [25], and its proof is similar to that in [25].

Theorem 2.1. *Let the root* $\mathbf{v}$ *of a tree* T *be an interior vertex. Let* T_1 *and* T_2 *be the complementary subtrees of* T*. Then with the same orientation of the graph and the subgraphs' edges described above,*

$$\phi_N(z) = \phi_N^{(1)}(z)\phi_D^{(2)}(z) + \phi_D^{(1)}(z)\phi_N^{(2)}(z), \qquad \phi_D(z) = \phi_D^{(1)}(z)\phi_D^{(2)}(z). \tag{2.34}$$

Proof. Fix the edges $e_j \in T_1$ and $e_r \in T_2$ incident with the root $\mathbf{v}$. (The case when $\mathbf{v}$ is a boundary vertex is even simpler.) Then the characteristic matrix $\Phi(z)$ can be expressed as

$$\Phi(z) = \left[\begin{array}{cccc|cccc}
* & \cdots & \cdots & * & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & \cdots & \cdots & * & 0 & \cdots & \cdots & 0 \\
0 & \cdots & l_{j,1}^0 R_{2n_{j,1}}^{(j,1)}\left(z, l_{j,1}^0\right) & R_{2n_{j,1}}^{(j,1)}(z,\infty) & -l_{j,2}^0 R_{2n_{r,2}}^{(r,2)}\left(z, l_{j,2}^0\right) & -R_{2n_{(r,2)}}^{(r,2)}(z,\infty) & \cdots & 0 \\
\hline
* & \cdots & l_{j,1}^0 R_{2n_{j,1}-1}^{(j,1)}\left(z, l_{j,1}^0\right) & R_{2n_{j,1}-1}^{(j,1)}(z,\infty) & l_{j,2}^0 R_{2n_{r,2}-1}^{(r,2)}\left(z, l_{j,2}^0\right) & R_{2n_{r,2}-1}^{(r,2)}(z,\infty) & \cdots & * \\
0 & \cdots & \cdots & 0 & * & \cdots & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & * & \cdots & \cdots & *
\end{array}\right].$$

Here the upper left square submatrix describes the continuity and Kirchhoff conditions at the vertices in T_1. So its determinant is $\phi_D^{(1)}(z)$, since the last row demonstrates the Dirichlet condition at $\mathbf{v}$. The lower right square submatrix describes those conditions in T_2 with the

Neumann boundary condition at **v**. So its determinant is $\phi_N^{(2)}(z)$. What remains in det $\Phi(z)$ is the product of the determinants of the upper left submatrix and lower right submatrix formed by interchanging the middle two row vectors concerning the continuity and Kirchhoff conditions at **v**. Hence the overall characteristic function $\phi_N(z)$ is given by

$$\phi_N(z) = \det\Phi(z) = \phi_D^{(1)}(z)\phi_N^{(2)}(z) - \det\begin{bmatrix} * & \cdots & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & \cdots & \cdots & * \\ * & \cdots & l_{j,1}^0 R_{2n_{j,1}-1}^{(j,1)}\left(z, l_{j,1}^0\right) & R_{2n_{j,1}-1}^{(j,1)}(z,\infty) \end{bmatrix}$$

$$\cdot \det\begin{bmatrix} -l_{j,2}^0 R_{2n_{r,2}}^{(r,2)}\left(z, l_{j,2}^0\right) & -R_{2n_{r,2}}^{(r,2)}(z,\infty) & \cdots & 0 \\ * & \cdots & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & \cdots & \cdots & * \end{bmatrix}$$

$$= \phi_D^{(1)}(z)\phi_N^{(2)}(z) + \phi_N^{(1)}(z)\phi_D^{(2)}(z).$$

Equation (2.34) is evident from the definition of $\phi_D(z)$. □

Corollary 2.2. *Suppose a tree T rooted at* **v** *has $d(\mathbf{v})$ complementary subtrees T_i ($i = 1, 2, \ldots, d(\mathbf{v})$) each rooted at* **v**. *Let $\phi_N^{(i)}$ and $\phi_D^{(i)}$ denote the Neumann and the Dirichlet characteristic functions for T_i. If* **v** *is a pendant vertex for T_i, then*

$$\phi_N(z) = \sum_{i=1}^{d(\mathbf{v})} \phi_N^{(i)}(z) \prod_{i=1,k\neq i}^{d(\mathbf{v})} \phi_D^{(k)}(z), \tag{2.35}$$

$$\phi_D(z) = \prod_{i=1}^{d(v)} \phi_D^{(i)}(z). \tag{2.36}$$

Now consider a subtree T_j. Deleting **v** together with the edge e_j incident with it, we obtain a subtree T_j' (see figure 2). We choose the vertex $\mathbf{v}'$ adjacent to **v** in T as the root of the subtree T_j'. We denote by $\phi_D^{(\prime)}(z)$ and $\phi_N^{(\prime)}(z)$ the Dirichlet and Neumann characteristic polynomials for the subtree T_j' rooted at $\mathbf{v}'$. Now considering the original subtree T_j rooted at $\mathbf{v}'$ we separate it into two subtrees T_j' and one consisting of one edge e_j only. Then applying theorem 2.1 we obtain the following obvious result.

Corollary 2.3. *Denote by N_j' the number of point masses on the edge e_j. Then*

$$\phi_N^{(j)}(z) = R_{2N_j'-1}^{(j)}(\infty, z)\phi_D^{(\prime)}(z) + l_0' R_{2N_j'-1}^{(j)}(l_0', z)\phi_N^{(\prime)}(z), \tag{2.37}$$

$$\phi_D^{(j)}(z) = R_{2N_j'}^{(j)}(\infty, z)\phi_D^{(\prime)}(z) + l_0' R_{2N_j'}^{(j)}(l_0', z)\phi_N^{(\prime)}(z). \tag{2.38}$$

Another application of theorem 2.1 lies in understanding the function $\frac{\phi_D}{\phi_N}$. For this, we need the notion of a Nevanlinna function. It is also called the R-function [19] or Herglotz function, and its definition also varies. Below is the definition we use in this paper.

Definition 2.4. *A meromorphic function $f(z)$ is said to be a Nevanlinna function if:*

(i) it is analytic in the half-planes $\operatorname{Im} z > 0$ *and* $\operatorname{Im} z < 0$;
(ii) $f(\overline{z}) = \overline{f(z)}$ $(\operatorname{Im} z \neq 0)$;
(iii) $\operatorname{Im} z \operatorname{Im} f(z) \geqslant 0$ *for* $\operatorname{Im} z \neq 0$.

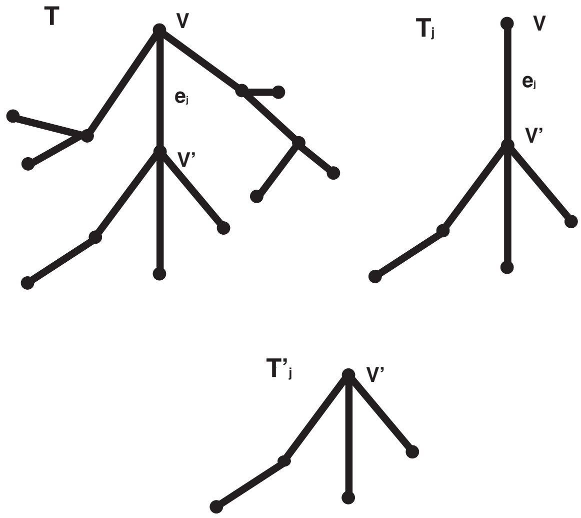

Figure 2.

Definition 2.5 (see [19])**.** *The Nevanlinna function $\omega(z)$ is said to be an S-function if $\omega(z) > 0$ for $z < 0$.*

Definition 2.6. *The S-function $\omega(z)$ is said to be an S_0-function if 0 is not a pole of $\omega(z)$.*

The following lemma is obvious.

Lemma 2.7. *Suppose that f and g are Nevanlinna functions; then $f + g$ and $-\frac{1}{f}$ are also Nevanlinna functions.*

Theorem 2.8. *The ratio*

$$\frac{\phi_D(z)}{\phi_N(z)}$$

is an S_0-function.

Proof. Substituting $z = \lambda^2$ into (2.8), we multiply it by $\overline{u_j^k}$. The imaginary part of the obtained equation is

$$\frac{\operatorname{Im}\left(\left(u_j^k - u_j^{k+1}\right)\overline{u_j^k}\right)}{l_j^k} - \frac{\operatorname{Im}\left(\left(u_j^{k-1} - u_j^k\right)\overline{u_j^{k-1}}\right)}{l_j^{k-1}} = \operatorname{Im} z\, m_j^k \left|u_j^k\right|^2. \tag{2.39}$$

Summing up (2.39) over k we obtain

$$\frac{\operatorname{Im}\left(\left(u_j^{n_j} - u_j^{n_j+1}\right)\overline{u_j^{n_j+1}}\right)}{l_j^{n_j}} - \frac{\operatorname{Im}\left(\left(u_j^0 - u_j^1\right)\overline{u_j^1}\right)}{l_j^0}$$
$$= \frac{\operatorname{Im}\left(\left(u_j^{n_j} - u_j^{n_j+1}\right)\overline{u_j^{n_j}}\right)}{l_j^{n_j}} - \frac{\operatorname{Im}\left(\left(u_j^0 - u_j^1\right)\overline{u_j^1}\right)}{l_j^j} = \operatorname{Im} z \sum_{k=1}^{n_j} m_j^k \left|u_j^k\right|^2. \tag{2.40}$$

Adding identities (2.40) over all edges and taking into account (2.9)–(2.11), (2.13), (2.14) we obtain

$$\mathrm{Im}\left(\overline{u}_1^{n_1+1}\sum_{j=1}^{d(\mathbf{v})}\frac{u_j^{n_j}-u_j^{n_j+1}}{l_j^{n_j}}\right)=\mathrm{Im}\, z\sum_{j=1}^{q}\sum_{k=1}^{n_j}m_j^k\left|u_j^k\right|^2,$$

where the sum on the left-hand side is taken over all edges incident with the root, and the sum on the right-hand side is taken over all the edges of the graph.

The last equation can be rewritten as

$$\mathrm{Im}\left(\frac{1}{u_1^{n_1+1}}\sum_{j=1}^{d(\mathbf{v})}\frac{u_j^{n_j}-u_j^{n_j+1}}{l_j^{n_j}}\right)=\frac{\mathrm{Im}\, z}{\left|u_1^{n_1+1}\right|^2}\sum_{j=1}^{q}\sum_{k=1}^{n_j}m_j^k\left|u_j^k\right|^2. \tag{2.41}$$

The right-hand side of (2.41) is positive. We note that $z=\lambda^2$ is a zero of $\sum_{j=1}^{d(\mathbf{v})}\frac{u_j^{n_j}-u_j^{n_j+1}}{l_j^{n_j}}$ if and only if λ is an eigenvalue of problem (2.8)–(2.11), (2.13), (2.14) and $z=\lambda^2$ is a zero of $u_1^{n_1+1}$ if and only if λ is an eigenvalue of problem (2.8)–(2.12). This means that

$$\frac{1}{u_1^{n_1+1}}\sum_{j=1}^{d(\mathbf{v})}\frac{u_j^{n_j}-u_j^{n_j+1}}{l_j^{n_j}}=-\frac{\phi_N(z)}{\phi_D(z)}.$$

Taking account of lemma 2.7, we conclude that $\frac{\phi_D(z)}{\phi_N(z)}$ is a Nevanlinna function. To finish the proof we note that according to (2.15)–(2.16) all $R_k^{(j)}(z)$ are positive for $z\in(-\infty,0]$ and consequently the polynomials $\phi_D(z)$ and $\phi_N(z)$ are positive for $z\in(-\infty,0]$. The theorem is proved. □

Corollary 2.9. *The zeros* $\left\{\nu_k^2\right\}$ *of* $\phi_D(z)$ *interlace with the zeros* $\left\{\mu_k^2\right\}$ *of* $\phi_N(z)$:

$$0<\mu_1^2\leqslant\nu_1^2\leqslant\mu_2^2\leqslant\nu_2^2\leqslant\cdots\leqslant\nu_N^2.$$

Proof. The ratio $\frac{\Phi_D(z)}{\Phi_N(z)}$ becomes an S_0-function after cancellation of common multipliers in the numerator and denominator, if any. The zeros of an S_0-function interlace in strict correspondence with its poles. The corollary is proved. □

Let $d(\mathbf{v})>1$. Then according to corollary 2.2

$$\frac{\phi_D(z)}{\phi_N(z)}=\frac{1}{\sum_{j=1}^{d(\mathbf{v})}\left(\frac{\phi_D^{(j)}(z)}{\phi_N^{(j)}(z)}\right)^{-1}}.$$

Let e_j be an edge incident with the root $\mathbf{v}$ and with a vertex $\mathbf{v}'\in T_j$. The number of point masses on this edge is denoted by N_j'. For this edge, the Lagrange identity (see [19]) written in our terms gives

$$R_{2N_j'}^{(j)}(l_{j0}',z)R_{2N_j'-1}^{(j)}(\infty,z)-R_{2N_j'-1}^{(j)}(l_{j0}',z)R_{2N_j'}^{(j)}(\infty,z)=-\frac{1}{l_{j0}'}. \tag{2.42}$$

Using (2.37), (2.38) and (2.42) we obtain

$$\frac{\phi_D^{(j)}(z)}{\phi_N^{(j)}(z)}=\frac{R_{2N_j'}^{(j)}(l_{j0}',z)}{R_{2N_j'-1}^{(j)}(l_{j0}',z)}+\frac{\phi_D^{(j)}(z)}{l_{j0}'\phi_N^{(j)}(z)R_{2N_j'-1}^{(j)}(l_{j0}',z)} \tag{2.43}$$

and

$$\frac{\phi_D^{(j)}(z)}{\phi_N^{(j)}(z)}=\frac{R_{2N_j'}^{(j)}(\infty,z)}{R_{2N_j'-1}^{(j)}(\infty,z)}-\frac{\phi_N^{(j)}(z)}{l_{j0}'\phi_N^{(j)}(z)R_{2N_j'-1}^{(j)}(\infty,z)}. \tag{2.44}$$

The following expansions in continued fractions can be found in [13]:

$$\frac{R^{(j)}_{2N'_j}(l'_{j0}, z)}{R^{(j)}_{2N'_j-1}(l'_{j0}, z)} = l'_{N'_j} + \cfrac{1}{-m'_{N'_j}z + \cfrac{1}{l'_{N'_j-1} + \cfrac{1}{-m'_{N'_j-1}z + \cdots + \cfrac{1}{l'_1 + \cfrac{1}{-m'_1 z + \cfrac{1}{l'_0}}}}}}, \qquad (2.45)$$

$$\frac{R^{(j)}_{2N'_j}(\infty, z)}{R^{(j)}_{2N'_j-1}(\infty, z)} = l'_{N'_j} + \cfrac{1}{-m'_{N'_j}z + \cfrac{1}{l'_{N'_j-1} + \cfrac{1}{-m'_{N'_j-1}z + \cdots + \cfrac{1}{l'_1 + \cfrac{1}{-m'_1 z}}}}}, \qquad (2.46)$$

$$l'_0 \frac{R^{(j)}_{2N'_j-1}(l'_0, z)}{R^{(j)}_{2N'_j-1}(\infty, z)} = l'_0 + \cfrac{1}{-m'_1 z + \cfrac{1}{l'_1 + \cfrac{1}{-m'_2 z + \cdots + \cfrac{1}{l'_{N'_j-1} + \cfrac{1}{-m'_{N'_j} z}}}}}, \qquad (2.47)$$

$$l'_0 \frac{R^{(j)}_{2N'_j}(l'_0, z)}{R^{(j)}_{2N'_j}(\infty, z)} = l'_0 + \cfrac{1}{-m'_1 z + \cfrac{1}{l'_1 + \cfrac{1}{-m'_2 z + \cdots + \cfrac{1}{l'_{N'_j-1} + \cfrac{1}{-m'_{N'_j} z + \cfrac{1}{l'_{N'_j}}}}}}}. \qquad (2.48)$$

Here l'_k are the subintervals and m'_k are the masses located on e_j.

An S_0-function $\frac{\phi^{(j)}_D(z)}{\phi^{(j)}_N(z)}$ with p_j ($p_j \leqslant N_j$) zeros and p_j poles can be expanded in the continued fraction,

$$\frac{\phi^{(j)}_D(z)}{\phi^{(j)}_N(z)} = a_{N'_j} + \cfrac{1}{-b_{N'_j}z + \cfrac{1}{a_{N'_j-1} + \cfrac{1}{-b_{N'_j-1}z + \cdots + \cfrac{1}{a_1 + \cfrac{1}{-b_1 z + \cfrac{1}{a_0 + f_1(z)}}}}}}, \qquad (2.49)$$

where

$$f_1(z) = \tilde{a}_0 + \cfrac{1}{-b_{N'_j+1}z + \cfrac{1}{a_{N'_j+1} + \cfrac{1}{-b_{N'_j+2}z + \cdots + \cfrac{1}{a_{p_j-1} + \cfrac{1}{-b_{p_j}z + \cfrac{1}{a_{p_j}}}}}}}.$$

Here $a_0 > 0$, $\tilde{a}_0 \geqslant 0$. We note that there is ambiguity in choice of a_0 and $\tilde{a}_0$. The expansion into the continued fractions gives only the value of the sum $a_0 + \tilde{a}_0$. However, we choose a_0 such that $\sum_{k=0}^{N'_j} a_k = l_j$. It will be clear below that this is possible, i.e., that $\tilde{a}_0 + \sum_{k=0}^{p_j} a_k > l_j$. Comparing (2.49) with (2.43) for $z \to \infty$ and taking into account (2.45) we conclude that $a_k = l'_k$ and $b_k = m'_k$ for each $k = 0, 1, 2, \ldots, N'_j$ and that $p_j > N'_j$. Now using (2.43), (2.44), (2.45) and (2.49) we see that the set of zeros of $f_1(z)$ coincides with the set of zeros of $\phi^{(\prime)}_D(z)$ while the set of poles of $f_1(z)$ coincides with the set of zeros of $\phi^{(\prime)}_N(z)$. This means that $f_1(z) = C\frac{\phi^{(\prime)}_D(z)}{\phi^{(\prime)}_N(z)}$ where C is a nonzero real constant.

Let us express $\frac{\phi_N(0)}{\phi_D(0)}$ via the lengths of the intervals l_j. First of all, corollary 2.2 implies

$$\frac{\phi_N(0)}{\phi_D(0)} = \sum_{j=1}^{d(\mathbf{v})} \frac{\phi^{(j)}_N(0)}{\phi^{(j)}_D(0)}.$$

Here for those of subtrees T_j which consist not just of one edge according to (2.37), (2.38) we have

$$\frac{\phi_N^{(j)}(0)}{\phi_D^{(j)}(0)} = \frac{R^{(j)}_{2N'_j-1}(\infty,0)\phi_D^{(\prime)}(0) + l'_0 R^{(j)}_{2N'_j-1}(l'_0,0)\phi_N^{(\prime)}(0)}{R^{(j)}_{2N'_j}(\infty,0)\phi_D^{(\prime)}(0) + R^{(j)}_{2N'_j}(l'_0,0)\phi_N^{(\prime)}(0)}$$
$$= \frac{R^{(j)}_{2N'_j-1}(l'_0,0)}{R^{(j)}_{2N'_j}(l'_0,0)} \frac{\frac{R^{(j)}_{2N'_j-1}(\infty,0)}{l'_0 R^{(j)}_{2N'_j-1}(l'_0,0)} + \frac{\phi_N^{(\prime)}(0)}{\phi_D^{(\prime)}(0)}}{\frac{R^{(j)}_{2N'_j}(\infty,0)}{l'_0 R^{(j)}_{2N'_j}(l'_0,0)} + \frac{\phi_N^{(\prime)}(0)}{\phi_D^{(\prime)}(0)}}. \tag{2.50}$$

Using (2.46), (2.47) and (2.48) we obtain from (2.50)

$$\frac{\phi_D^{(j)}(0)}{\phi_N^{(j)}(0)} = l_j + \frac{\phi_D^{(\prime)}(0)}{\phi_N^{(\prime)}(0)}. \tag{2.51}$$

On the other hand, (2.49) implies

$$\frac{\phi_D^{(j)}(0)}{\phi_N^{(j)}(0)} = l_j + f_1(0),$$

which gives $C = 1$ and therefore $f_1(z) = \frac{\phi_D^{(\prime)}(z)}{\phi_N^{(\prime)}(z)}$ and $f_1(0) > 0$, and consequently $\tilde{a}_0 + \sum_{k=0}^{p_j} a_k > l_j$.

Let us recall that T'_j is the subtree of T_j obtained from T_j by deleting the edge e_j incident with $\mathbf{v}$ and $\mathbf{v}'$ is the other vertex incident with e'_j. The subtree T'_j is rooted at $\mathbf{v}'$. Denote by $d(\mathbf{v}')$ the degree of $\mathbf{v}'$ as of the root of T'_j. Then

$$\frac{\phi_N^{(\prime)}(z)}{\phi_D^{(\prime)}(z)} = \sum_{k=1}^{d(\mathbf{v}')} \frac{\phi_N^{(k')}(z)}{\phi_D^{(k')}(z)},$$

where $\phi_N^{(k')}(z)$ and $\phi_D^{(k')}(z)$ are characteristic polynomials for Neumann and Dirichlet problems, correspondingly, on the complementary subtrees $T'_{j,k}$ $(k = 1, 2, \ldots, d(\mathbf{v}'))$ of T'_j rooted at $\mathbf{v}'$.

Thus, we can continue expanding into continuous fractions

$$f_1(z) = \frac{\phi_D^{(\prime)}(z)}{\phi_N^{(\prime)}(z)} = \frac{1}{\sum_{k=1}^{d(\mathbf{v}')} \frac{\phi_N^{(k')}(z)}{\phi_D^{(k')}(z)}} = \frac{1}{\sum_{k=1}^{d(\mathbf{v}')} \left(\tilde{a}_0^{(k')} + \cfrac{1}{-b^{(k')}_{N'+1} z + \cfrac{1}{a^{(k')}_{N'+1} + \cfrac{1}{-b^{(k')}_{N'+2} z + \cdots + \cfrac{1}{a^{(k')}_{N-1} + \cfrac{1}{-b^{(k')}_{N} z + \frac{\phi_D^{(k'')}(z)}{\phi_N^{(k'')}(z)}}}}}} \right)}. \tag{2.52}$$

Here $\phi_N^{(k'')}(z)$ and $\phi_D^{(k'')}(z)$ are characteristic polynomials for Neumann and Dirichlet problems, correspondingly, on the complementary subtrees of T''_k rooted at $\mathbf{v}''$ where T''_k are the subtrees obtained from T'_j in the way T'_j was obtained from T. This expansion into 'branching' continued fractions can be continued.

Note that (2.51) is a recurrence formula which together with the conditions $\frac{\phi_N^{(jr)}(0)}{\phi_D^{(jr)}(0)} = \frac{1}{l_{jr}}$ for pendant edges e_{jr} determines $\frac{\phi_N^{(j)}(0)}{\phi_D^{(j)}(0)}$ and $\frac{\phi_N(0)}{\phi_D(0)}$. Thus, we can find $\frac{\phi_N(0)}{\phi_D(0)}$ and the analogous quantity for each subtree and these quantities are independent of mass distribution. We call it the form characteristic of the subtree T for which $\mathbf{v}$ is the root and we denote it by $\Phi_{T,\mathbf{v}}$.

The proofs of the following theorems 2.10 and 2.13 and corollaries 2.11, 2.12, 2.14 are nearly the same as in [25] for the continuous case, therefore we omit them.

Theorem 2.10. *Denote by $p_N(z)$ the multiplicity of z as a zero of ϕ_N and by $p_D(z)$ the multiplicity of z as a zero of $\phi_D(z)$. Then*

(a) $|p_N(z) - p_D(z)| \leqslant 1$;
(b) $p_N(z) \leqslant q - \kappa$, *where κ is the number of interior vertices.*

Corollary 2.11.

(a) If the root $\mathbf{v}$ is an interior vertex then $p_D(z) \leqslant q - \kappa + 1$;
(b) If $\mathbf{v}$ is a pendant vertex then

$$p_D(z) + p_N(z) \leqslant 2q - 2\kappa - 1.$$

Corollary 2.12. *Let the root be an interior vertex. Then*

(i) The multiplicity of any eigenvalue of problem (2.8)–(2.11), (2.13), (2.14) does not exceed $q - \kappa$.
(ii) The multiplicity of any eigenvalue of problem (2.8)–(2.12) does not exceed $q - \kappa + 1$.

Theorem 2.13. *Suppose that z_0 is a common zero of $\phi_N(z)$ and $\phi_D(z)$ with multiplicities p_N and p_D, respectively. Then $p_N \geqslant p_D$ implies that $p_D \leqslant q - d(\mathbf{v}) - \kappa + 1$.*

Let us consider a star-shaped graph of q edges rooted at the interior vertex.
Denote by $\{\mu_k^2\}$ the set of zeros of $\phi_N(z)$ and by $\{\nu_k^2\}$ the set of zeros of $\phi_D(z)$.

Corollary 2.14. *For a star-shaped graph rooted at the interior vertex*

(a) $\nu_k^2 = \mu_{k+1}^2$ *if and only if* $\mu_{k+1}^2 = \nu_{k+1}^2$.
(b) Multiplicity of ν_k^2 does not exceed q.

Thus, we have obtained theorem 2.2 in [3] as a particular case.

3. Inverse problem for a tree of Stieltjes strings

Here we give a positive answer to the question: given two sequences of interlacing real nonzero numbers symmetric with respect to the origin, a rooted metric tree of a prescribed form with prescribed lengths of edges, does a distribution of point masses and subintervals on the edges exist such that the two sequences are Neumann and Dirichlet spectra for this tree?

Theorem 3.1. *Let $\{\mu_k\}_{k=-n,k\neq 0}^{n}$ and $\{\nu_k\}_{k=-n,k\neq 0}^{n}$ be symmetric ($\mu_{-k} = -\mu_k, \nu_{-k} = -\nu_k$) and monotonic sequences of real numbers which interlace:*

$$0 < (\mu_1)^2 < (\nu_1)^2 < \cdots < (\mu_n)^2 < (\nu_n)^2.$$

Let T be a metric tree of a prescribed form rooted at a vertex $\mathbf{v}$ with complementary subtrees T_j ($j = 1, 2, \ldots, d(\mathbf{v})$ where $d(\mathbf{v})$ is the degree of the root) with prescribed lengths of edges l_j ($j = 1, 2, \ldots, q$, q is the number of edges in T). Let N_j ($j = 1, 2, \ldots, d(\mathbf{v})$) be nonnegative integers and $\sum_{j=1}^{d(\mathbf{v})} N_j = n$.

Then there exist sets of positive numbers $\{m_k^j\}_{k=1}^{n_j}$ (point masses on the edge e_j, $j = 1, 2, \ldots, q$) and nonnegative numbers $\{l_k^j\}_{k=0}^{n_j}$ (lengths of subintervals on the edge e_j) such *that* $\sum_{k=0}^{n_j} l_k^j = l_j$, $\sum_{j=1}^{q} n_j = n$, *the numbers of point masses on the complementary subtrees* T_j *are* N_j $(j = 1, 2, \ldots, d(\mathbf{v}))$ *and the spectrum of the Neumann problem* (2.8)–(2.11), (2.13), (2.14) *coincides with* $\{\mu_k\}_{k=-n,k\neq 0}^{n}$, *and the spectrum of the Dirichlet problem* (2.8)–(2.12) *coincides with* $\{\nu_k\}_{k=-n,k\neq 0}^{n}$.

Proof. Let us use induction by the length of the longest path starting at the root. If the length of the longest path is 1, we are dealing with a star graph and the statement of our theorem has been proved in [3]. Let m denote the maximal length of path in T starting at the root. Let us assume the statement of the theorem to be true for each rooted tree with $m = p - 1$ and let us prove it for a tree with $m = p$.

First of all, we consider the rational function

$$\Phi_{T,\mathbf{v}} \prod_{k=1}^{n} \frac{1 - \frac{z}{\mu_k^2}}{1 - \frac{z}{\nu_k^2}}, \tag{3.1}$$

where $\Phi_{T,\mathbf{v}} > 0$ is the form characteristic of the tree. Here $\Phi_{T,\mathbf{v}}$ is known because the form of T is given as well as the length of each edge.

We will show that (3.2) is nothing but $\frac{\phi_N(z)}{\phi_D(z)}$ for our given graph and a certain distribution of masses and a certain division of the edges into subintervals.

If $d(\mathbf{v}) > 1$ then we divide the tree into $d(\mathbf{v})$ complementary subtrees T_j each rooted at $\mathbf{v}$. If the root is pendant this step is unnecessary. It is clear that $\left(\Phi_{T,\mathbf{v}} \prod_{k=1}^{n} \frac{1-\frac{z}{\mu_k^2}}{1-\frac{z}{\nu_k^2}}\right)^{-1}$ belongs to S_0 and therefore

$$\Phi_{T,\mathbf{v}} \prod_{k=1}^{n} \frac{1 - \frac{z}{\mu_k^2}}{1 - \frac{z}{\nu_k^2}} = \sum_{k=1}^{n} \frac{A_k}{z - \nu_k^2} + B,$$

where $A_k > 0$ and $B = \Phi_{T,\mathbf{v}} + \sum_{k=1}^{n} \frac{A_k}{\nu_k^2}$. We arrange the set $\{\nu_k^2\}_{k=1}^{n}$ as the union of disjoint sets $\left\{\nu_{k_s^{(1)}}^2\right\}_{s=1}^{N_1}, \left\{\nu_{k_s^{(2)}}^2\right\}_{s=1}^{N_2}, \ldots, \left\{\nu_{k_s^{d(\mathbf{v})}}^2\right\}_{s=1}^{N_{d(\mathbf{v})}}$. Then $\sum_{j=1}^{d(\mathbf{v})} N_j = n$ and

$$\Phi_{T,\mathbf{v}} \prod_{k=1}^{n} \frac{1 - \frac{z}{\mu_k^2}}{1 - \frac{z}{\nu_k^2}} = \sum_{j=1}^{d(\mathbf{v})} \left(\sum_{s=1}^{N_j} \frac{A_{k_s}}{z - \nu_{k_s}^2} + B_j \right),$$

where we choose B_j such that

$$B_j = \Phi_{T_j,\mathbf{v}} + \sum_{s=1}^{N_j} \frac{A_{k_s}}{\nu_{k_s}^2}, \tag{3.2}$$

and $\Phi_{T_j,\mathbf{v}}$ are the form characteristics of the subtrees T_j and consequently $\Phi_{T,\mathbf{v}} = \sum_{j=1}^{d(\mathbf{v})} \Phi_{T_j,\mathbf{v}}$.

The rational function

$$\sum_{s=1}^{N_j} \frac{A_{k_s}}{z - \nu_{k_s}^2} + B_j$$

has N_j simple zeros and N_j simple poles and due to inequalities $A_k > 0$ and to (3.3) we conclude that

$$\left(\sum_{s=1}^{N_j} \frac{A_{k_s}}{z - \nu_{k_s}^2} + B_j \right)^{-1}$$

belongs to S_0 for each $j = 1, 2, \ldots, d(\mathbf{v})$. We will prove that these rational functions are $\frac{\phi_N^{(j)}(z)}{\phi_D^{(j)}(z)}$, where $\phi_N^{(j)}(z)$ and $\phi_D^{(j)}(z)$ are the characteristic polynomials for the Neumann and

Dirichlet problems on the subtree T_j for a certain distribution $\{m_k^j\}_{k=1}^{n_j}$ $(j = 1, 2, \ldots, q)$ of point masses and a certain distribution of subintervals $\{l_k^j\}_{k=0}^{n_j}$ $\left(\sum_{k=0}^{n_j} l_k^j = l_j\right)$ on the edges of T_j.

Let us expand $\left(\sum_{s=1}^{N_j} \frac{A_{k_s}}{z-\nu_{k_s}^2} + B_j\right)^{-1}$ into continued fractions of the form

$$\left(\sum_{s=1}^{N_j} \frac{A_{k_s}}{z-\nu_{k_s}^2} + B_j\right)^{-1} = a_{N_j'} + \cfrac{1}{-b_{N_j'}z + \cfrac{1}{a_{N_j'-1} + \cfrac{1}{-b_{N_j'-1}z + \cdots + \cfrac{1}{a_1 + \cfrac{1}{-b_1 z + \frac{1}{a_0}}}}}} + \frac{Q_D(z)}{P_D(z)},$$

$$\left(\sum_{s=1}^{N_j} \frac{A_{k_s}}{z-\nu_{k_s}^2} + B_j\right)^{-1} = a_{N_j'} + \cfrac{1}{-b_{N_j'}z + \cfrac{1}{a_{N_j'-1} + \cfrac{1}{-b_{N'-1}z + \cdots + \cfrac{1}{a_1 + \frac{1}{-b_1 z}}}}} + \frac{Q_N(z)}{P_N(z)}. \tag{3.3}$$

Here N_j' is defined by the condition $\sum_{k=0}^{N_j'} a_k = l_j$. Polynomials $Q_N(z)$ and $Q_D(z)$ are of degree $N_j - N_j'$, and $P_N(z)$ and $P_D(z)$ are polynomials of degree $N_j + N_j'$. We identify a_k-s with the lengths of subintervals and b_k-s with point masses on the edge e_j.

Let us prove that $\frac{Q_D(z)}{Q_N(z)}$ is an S_0-function. To do this, let us note that

$$\left(\sum_{s=1}^{N_j} \frac{A_{k_s}}{z-\nu_{k_s}^2} + B_j\right)^{-1} = a_{N_j'} + \cfrac{1}{-b_{N_j'}z + \cfrac{1}{a_{N_j'-1} + \cfrac{1}{-b_{N_j'-1}z + \cdots + \cfrac{1}{a_1 + \cfrac{1}{-b_1 z + \frac{1}{a_0 + f_1(z)}}}}}}, \tag{3.4}$$

where

$$f_1(z) = \tilde{a}_0 + \cfrac{1}{-b_{N_j'+1}z + \cfrac{1}{a_{N_j'+1} + \cfrac{1}{-b_{N_j'+2}z + \cdots + \cfrac{1}{a_{N_j-1} + \cfrac{1}{-b_{N_j}z + \frac{1}{a_{N_j}}}}}}}. \tag{3.5}$$

Comparing (3.3) with (3.4) we note that the zeros of $Q_D(z)$ are nothing but the zeros of $f_1(z)$, and the zeros of $Q_N(z)$ are the zeros of the function $f_2(z) \stackrel{\text{def}}{=} \frac{1}{a_0+f_1(z)}$ or, which is the same, the poles of $f_1(z)$. Judging by (3.5) we conclude that $f_1(z)$ is an S_0-function.

Substituting $z = 0$ into (3.4) and making use of (3.3) we obtain

$$\Phi_{T_j,\mathbf{v}}^{-1} = l_j + f_1(0).$$

On the other hand (2.51) means nothing but

$$\Phi_{T_j,\mathbf{v}}^{-1} = l_j + \Phi_{T_j',\mathbf{v}'}^{-1}$$

and consequently $f_1(0) = \Phi_{T_j',\mathbf{v}'}^{-1}$.

The function $f_1(z)$ belongs to S_0 and has $N_j - N_j'$ zeros and $N_j - N_j'$ poles and the length of the longest path in T_j' starting at $\mathbf{v}'$ is equal to $p - 1$. The corresponding distribution of masses and subintervals on T_j' exists according to the induction assumption. The theorem is proved. □

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